Math 2040 C Week 4

Dimension

Given V, want to define

dimension = number of elements in a basis of V

Q Well-defined?

> **Prop 2.35** Let V be finite dim.
> All bases of V are finite and have same number of elements

Pf By Thm 2.32, V has a finite basis S

Suppose S' is another basis, then

S' is linear indept and span $S = V$

$\Rightarrow$ S' is finite and $|S'| \leq |S|$

Similarly, $|S| \leq |S'|$ $\Rightarrow$ $|S'| = |S|$

Hence, we can define

> **Defn** Let V be a finite dim vector space.
> Then the dimension of V is defined to be
> $$\dim V = |S|$$
> where S is any basis of V

eg Let $V = \mathbb{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$

As a complex vector space,

V has a basis $\{(1,0), (0,1)\}$

As a real vector space,

V has a basis $\{(1,0), (i,0), (0,1), (0,i)\}$

$\therefore$ V has complex dim 2, real dim 4

eg ① $\dim \mathbb{F}^n = n$ ② $\dim P_m(\mathbb{F}) = m+1$

③ $\dim M_{m \times n}(\mathbb{F}) = mn$

④ $\dim \{A \in M_{n \times n}(\mathbb{F}) : A^t = A\} = \frac{n(n+1)}{2}$

Prop 2.38

If V is finite dim with subspace $U \subseteq V$ then $\dim U \leq \dim V$

Pf V is finite dim

$\Rightarrow U$ is finite dim (Prop 2.26)

Let S be a basis of U.

Then S is lin. indept.

$\Rightarrow \exists$ a basis S' of V with $S \subseteq S'$ (Prop 2.33)

$\therefore \dim U = |S| \leq |S'| = \dim V$

Prop 2.39 and 2.42

Suppose $\dim V = n$ and $|S| = n$. If either

① S is lin indept or

② $\operatorname{span} S = V$

Then S is a basis of V

Pf ① Suppose S is lin indept. Then by Prop 2.23

$\exists$ a basis S' of V with $S \subseteq S'$

$\dim V = n \Rightarrow |S'| = n = |S|$

$\Rightarrow S = S'$ is a basis of V

② Suppose $\operatorname{span} S = V$. Then by Prop 2.31

$\exists$ a basis S' of V with $S' \subseteq S$

$\dim V = n \Rightarrow |S'| = n = |S|$

$\Rightarrow S = S'$ is a basis of V

eg $(1,i), (1,-i) \in \mathbb{C}^2$ are lin. indept over $\mathbb{C}$

Also, $\dim \mathbb{C}^2 = 2$

$\Rightarrow \{(1,i),(1,-i)\}$ is a basis of $\mathbb{C}^2$

eg Find a basis for $V=\{p(x)\in P_3(\mathbb{R}) : p'(2)=0\}$

Sol Method 1:

Let $p(x)=a+bx+cx^2+dx^3\in P_3(\mathbb{R})$

Then $p'(x)=b+2cx+3dx^2$

$\therefore\ p(x)\in V \iff p'(2)=b+4c+12d=0$

$$\left[\begin{array}{cccc|c} 0 & \boxed{1} & 4 & 12 & 0\end{array}\right]$$

free variable

$$\begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix} a\\ -4c-12d\\ c\\ d\end{bmatrix}=a\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+c\begin{bmatrix}0\\-4\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\-12\\0\\1\end{bmatrix}$$

$\therefore\ \{1,-4x+x^2,-12x+x^3\}$ is a basis for V

Method 2:

(HW)

Note $x\notin V \Rightarrow V\subsetneq P_3(\mathbb{R}) \Rightarrow \dim V<\dim P_3(\mathbb{R})=4$

Also, $1,(x-2)^2,(x-2)^3\in V$ are lin indept $\Rightarrow \dim V\geq 3$.

$\therefore\ \dim V=3$. By last Prop

$\{1,(x-2)^2,(x-2)^3\}$ is a basis of V

Prop 2.43 Let V be finite dim

Let U_1,U_2 be subspaces of V. Then

$\dim(U_1+U_2)=\dim U_1+\dim U_2-\dim(U_1\cap U_2)$

Pf V is finite dim

$\Rightarrow U_1,U_2,U_1\cap U_2$ are finite dim. by Prop 2.26

Let $\{u_1,\cdots,u_m\}$ be a basis of $U_1\cap U_2$

Extend it to basis

$\{u_1,\cdots,u_m,v_1,\cdots,v_j\}$ of U_1

$\{u_1,\cdots,u_m,w_1,\cdots,w_k\}$ of U_2

One can show that U_1+U_2 has a basis

$\{u_1,\cdots,u_m,v_1,\cdots,v_j,w_1,\cdots,w_k\}$ (Exercise)

Hence, $\dim U_1+\dim U_2-\dim(U_1\cap U_2)$

$=(m+j)+(m+k)-m$

$=m+j+k$

$=\dim(U+V)$

Linear Maps

Vector spaces are sets with linear structure $(+, \cdot)$

Want maps between them to preserve this structure:

We assume V, W are vector spaces over $\mathbb{F}$

Sometimes, we write $0_V, 0_W$ for the zero vector of V, W respectively

Defn 3.2

A function $T: V \to W$ is called a linear map if

1. Additivity: $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$
2. Homogeneity: $T(\lambda v) = \lambda T(v) \quad \forall v \in V, \lambda \in \mathbb{F}$

Rmk Another notation $T(v) = Tv$ (especially when v is a function)

Another name for linear map is linear transformation

Notation 3.3

$\mathcal{L}(V, W)$ = the set of all linear maps from V to W

Examples

1. Zero map $T_0 : V \to W$

$$T_0(v) = 0_W \quad \forall v \in V$$

Rmk Our book denotes a zero map by $0: V \to W$

2. Identity map I or $I_V : V \to V$

$$I(v) = v \quad \forall v \in V$$

3. Differentiation $D: V \to V$ for $V = \mathcal{P}(\mathbb{R})$ or $\mathcal{P}_m(\mathbb{R})$

or $\mathcal{C}^\infty(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} : f^{(k)}(x) \text{ exists } \forall x \in \mathbb{R}, k \geq 0 \}$

$$D(f) = f' \quad \forall f \in V$$

4. Integration $T: \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ ← a real vector space of dim 1

$$T(p(x)) = \int_0^1 p(x)\, dx$$

5. Multiplication $T: \mathbb{R}^{(0,1)} \to \mathbb{R}^{(0,1)}$

$$(Tf)(x) = e^x f(x)$$

⑥ <u>Shifting</u> $T: \mathbb{F}^\infty \to \mathbb{F}^\infty$

Backward shift: $T(a_1, a_2, a_3, \cdots) = (a_2, a_3, \cdots)$

Forward shift: $S(a_1, a_2, a_3, \cdots) = (0, a_1, a_2, \cdots)$

<u>eg</u> Verify ⑤ is a linear map:

Let $f, g \in \mathbb{R}^{(0,1)}$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}[T(f+g)](x) &= e^x (f+g)(x) \\ &= e^x [f(x) + g(x)] \\ &= e^x f(x) + e^x g(x) \\ &= (Tf)(x) + (Tg)(x) \\ &= (Tf + Tg)(x)\end{aligned}$$

$\Rightarrow T(f+g) = Tf + Tg$

Similarly

$$\begin{aligned}[T(\lambda f)](x) &= e^x (\lambda f)(x) \\ &= e^x [\lambda f(x)] \\ &= \lambda [e^x f(x)] \\ &= \lambda [Tf(x)] \\ &= [\lambda (Tf)](x)\end{aligned}$$

$\Rightarrow T(\lambda f) = \lambda (Tf)$

$\therefore T$ is linear

$\boxed{\text{Ex Verify others}}$

Prop If $T: V \to W$ is linear, then

① $T(0_V) = 0_W$

② $T(\sum c_i v_i) = \sum c_i T(v_i) \quad \forall c_i \in \mathbb{F}, v_i \in V$

<u>Pf</u> ① $T(0_V) = T(\underset{\substack{\uparrow \\ \text{zero in } \mathbb{F}}}{0} \cdot 0_V) = 0\, \underbrace{T(0_V)}_{\in W} = 0_W$

② $T(\sum c_i v_i) \underset{\substack{\uparrow \\ \text{Additivity + Induction}}}{=} \sum T(c_i v_i) \underset{\substack{\uparrow \\ \text{Homogeneity}}}{=} \sum c_i T(v_i)$

eg Are they well-defined linear map?

① $T: \mathbb{C}^3 \to \mathbb{C}^2$, $T(z_1,z_2,z_3) = (z_1+z_2, z_3+2)$

② $T: \mathbb{F}^\infty \to \mathbb{F}^\infty$, $T(a_1,a_2,a_3,\cdots) = (1,a_1,a_2,\cdots)$

③ $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$, $T(p(x)) = \int_1^x p(t)\,dt$

④ $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x_1,x_2) = (x_1^2, x_2^2)$

⑤ $T: \mathbb{C}^2 \to \mathbb{C}^2$, $T(z_1,z_2) = (z_1, \overline{z_2})$

Sol We do ①, ③, ⑤

① Let $\vec{u} = (1,0,0)$ $v = (0,1,0)$

$T(\vec{u}+\vec{v}) = T(1,1,0) = (2,2)$

$T(\vec{u}) + T(\vec{v}) = (1,2)+(1,2) = (2,4)$

$\therefore T(\vec{u}+\vec{v}) \neq T(\vec{u}) + T(\vec{v})$

$\Rightarrow$ T is not linear

Alt soln $T(0,0,0) = (0,2) \neq (0,0)$

$\therefore$ T is not linear by last Prop

③ $T(x^3) = \int_1^x t^3\,dt = \left[\frac{1}{4}t^4\right]_1^x = \frac{1}{4}x^4 - \frac{1}{4} \notin P_3(\mathbb{R})$

The map is not even well-defined

Rmk $T: P(\mathbb{R}) \to P(\mathbb{R})$ with same formula is a well-defined linear map.

⑤ Answer depends on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

- If $\mathbb{F} = \mathbb{C}$, then let $v = (1,2)$

  $i\,T(v) = i\,T(1,2) = i(1,2) = (i, 2i)$

  $T(iv) = T(i, 2i) = (i, -2i) \neq i\,T(v)$

  $\therefore$ T is not (complex) linear

- If $\mathbb{F} = \mathbb{R}$, then $\forall z = (z_1,z_2), w = (w_1,w_2) \in \mathbb{C}^2$, $\lambda \in \mathbb{R}$

$$\begin{aligned} T(z+w) &= T(z_1+w_1, z_2+w_2) \\ &= (z_1+w_1, \overline{z_2+w_2}) \\ &= (z_1, \overline{z_2}) + (w_1, \overline{w_2}) \\ &= T(z) + T(w) \end{aligned}$$

$T(\lambda z) = T(\lambda z_1, \lambda z_2) = (\lambda z_1, \overline{\lambda z_2}) = \lambda(z_1, \overline{z_2}) = \lambda T(z)$

$\therefore$ T is (real) linear

Prop 3.5

Suppose $S=\{v_1,\cdots,v_n\}$ is a basis for V and $w_1,\cdots,w_n\in W$

Then $\exists$ unique linear map $T:V\to W$ such that $T(v_i)=w_i \quad \forall i=1,\cdots,n$

Rmk A linear map is uniquely determined by the images of a basis (even if $\dim V=\infty$)

Pf We prove it in 4 steps

① Define T

We first define T. Since S is basis of V, for any $v\in V$, $\exists$ unique $c_1,\cdots,c_n\in\mathbb{F}$ s.t.

$$v=c_1v_1+\cdots+c_nv_n$$

Define $T(v)=c_1w_1+\cdots+c_nw_n$

c_i's are uniquely chosen $\Rightarrow$ T is well-defined

② Show that T is linear

Suppose $\lambda\in\mathbb{F}$ and $u,v\in V$ with $u=a_1v_1+\cdots+a_nv_n$, $v=b_1v_1+\cdots+b_nv_n$

Then
$$\begin{aligned}T(u+v)&=T\Big(\sum_i a_iv_i+\sum_i b_iv_i\Big)\\&=T\Big(\sum_i(a_i+b_i)v_i\Big)\\&=\sum_i(a_i+b_i)w_i\\&=\sum_i a_iw_i+\sum_i b_iw_i\\&=T(u)+T(v)\end{aligned}$$

Similarly
$$\begin{aligned}T(\lambda u)&=T\Big(\lambda\sum a_iv_i\Big)\\&=T\Big(\sum(\lambda a_i)v_i\Big)\\&=\sum(\lambda a_i)w_i\\&=\lambda\Big(\sum a_iw_i\Big)\\&=\lambda T(u)\end{aligned}$$

$\therefore$ T is linear

③ Show $T(v_i) = w_i$

$$T(v_1) = T(1v_1 + 0v_2 + \cdots + 0v_n)$$
$$= 1w_1 + 0w_2 + \cdots + 0w_n$$
$$= w_1$$

Similarly $T(v_i) = w_i \quad \forall i = 1, \cdots, n$

④ Show uniqueness of T

Suppose $\tilde{T}: V \to W$ is linear with

$$\tilde{T}(v_i) = w_i \quad \forall i = 1, \cdots, n$$

Then for any $c_1, \cdots, c_n \in \mathbb{F}$

$$\tilde{T}(\sum c_i v_i) = \sum c_i \tilde{T}(v_i)$$
$$= \sum c_i w_i$$
$$= T(\sum c_i v_i)$$

$\operatorname{span} S = V \Rightarrow \tilde{T}(v) = T(v) \ \forall v \in V$

$\Rightarrow \tilde{T} = T$

$\therefore$ T is unique.

Operations on linear Maps

Defn 3.6 Let $S, T \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}$.
Define $S+T, \lambda T \in \mathcal{L}(V, W)$ by

$$(S+T)(v) = S(v) + T(v) \ \ \& \ \ (\lambda T)(v) = \lambda(T(v))$$

for any $v \in V$

Ex Verify that $S+T$ and λT are really linear:

Prop 3.7
$\mathcal{L}(V, W)$ with operations above is a vector space.

Pf Exercise

Rmk The zero transformation $T_0 : V \to W$ is the zero vector in $\mathcal{L}(V, W)$ (Recall: $T_0(v) = 0_W \ \forall v \in V$)

For $T \in \mathcal{L}(V, W)$, $\underbrace{(-T)}_{\text{additive inverse in } \mathcal{L}(V,W)}(v) = \underbrace{-T(v)}_{\text{in } W} \quad \forall v \in V$

Prop Let U, V, W be vector spaces over $\mathbb{F}$. If $T \in L(U,V)$, $S \in L(V,W)$, then $S \circ T \in L(U,W)$

Pf Let $u, v \in U$, then

$$(S\circ T)(u+v) = S(T(u+v)) = S(T(u)+T(v))$$
$$= S(T(u)) + S(T(v))$$
$$= (S\circ T)(u) + (S\circ T)(v)$$

Similarly, $(S\circ T)(\lambda v) = \lambda[(S\circ T)(v)]$ for $\lambda \in \mathbb{F}$

Notation 3.8 We write $ST = S\circ T$

ST is called the product of S and T.

Rmk Take $U = V = W$. Then composition defines a product structure on $L(U,U)$:

$$S, T \in L(U,U) \xrightarrow[\text{Composition}]{} ST \in L(U,U)$$

Warning: This multiplication is not commutative

i.e. $ST \neq TS$ in general

(Similar to matrix: $AB \neq BA$)

Prop 3.9 Properties of sum and Product in $L(V,W)$

Associativity For $T_i \in L(V_i, V_{i+1})$, $i = 1,2,3$

$$(T_3 T_2) T_1 = T_3 (T_2 T_1)$$

Identity Let $I_V \in L(V,V)$, $I_W \in L(W,W)$ be identity map and $T \in L(V,W)$. Then

$$T I_V = I_W T = T$$

Distributive Property

For $T, T_1, T_2 \in L(U,V)$ $S, S_1, S_2 \in L(V,W)$

$$(S_1 + S_2)T = S_1 T + S_2 T$$
$$S(T_1 + T_2) = ST_1 + ST_2$$

$L(V,V)$ has nice addition and multiplication structures